Summation 2

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Arithmetic Progression (AP) :-

$a, a+d, a+2d, a+3d, \ldots, a+nd$

$$S = a + (a+d) + (a+2d) + \cdots + a+nd$$
$$S = (a+nd) + (a+(n-1)d) + \cdots + a$$
$$2S = (2a+nd) + (2a+nd) + \cdots + (2a+nd)$$
$$S = \frac{(n+1)(2a+nd)}{2} = \frac{1}{2}(n+1)(a+(a+nd)) = \frac{1}{2}(\text{no. of term})(\text{first term} + \text{last term})$$

Geometric Progression (GP) :-

$a, ar, ar^2, ar^3, \ldots, ar^n \quad (r \neq 1)$

$$S = a + ar + ar^2 + \cdots + ar^n$$
$$rS = ar + ar^2 + \cdots + ar^n + ar^{n+1}$$
$$(1-r)S = a - ar^{n+1}$$
$$S = \frac{a(1-r^{n+1})}{1-r}$$

If $|r| < 1$ then,

for $n \to \infty$ we get, $r^{n+1} \to 0$

then, $S = \frac{a(1-0)}{1-r} = \frac{a}{1-r}$

Lemma :- $(\varphi * 1 = id)$

Let $n \geq 1$ be an integer. Then,

$$\sum_{d|n} \varphi(d) = n$$

$\longrightarrow$ euler's totient function

$\varphi(n) = k$ means n has k integers less than it that are coprime to it (1 included)

Proof :- (later)

Q> Let n be a positive integer. Prove that $\lfloor \cdot \rfloor$

Q> Let n be a positive integer. Prove that

$$\sum_{k\geq 1} \varphi(k)\left\lfloor \frac{n}{k} \right\rfloor = \frac{1}{2}n(n+1)$$

$\lfloor \cdot \rfloor$
↳ floor function
$\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Ans:- The main idea is to rewrite the $\lfloor \frac{n}{k} \rfloor$ in terms of a summation involving divisors

If $n = kc + k'$, $k' < k$, $k' \geq 0$

Then, $\left\lfloor \frac{n}{k} \right\rfloor = c$ ↳ maximum count of number of k's we can add till the sum is less than or equal to n

$$\underbrace{k+k+\cdots\cdots+k}_{c} \leq n$$

$$\underbrace{k+k+\cdots+k}_{c+1} > n$$

$$\left\lfloor \frac{n}{k} \right\rfloor = \sum_{\substack{m\leq n \\ k|m}} (1) = c$$

$$\Rightarrow \sum_{k\geq 1} \varphi(k)\left\lfloor \frac{n}{k} \right\rfloor = \sum_{k\geq 1}\Big(\varphi(k)\sum_{\substack{m\leq n \\ k|m}} 1\Big)$$

$$= \sum_{k\geq 1}\sum_{\substack{m\leq n \\ k|m}} \varphi(k)$$

We are computing the summation of $\varphi(k)$ over all k and $m \leq n$ with $k|m$.

First summation selects k and runs over it and second summation selects m and runs over it with $k|m$ condition

So we can swap as below:-

$$\sum_{m=1}^{n}\sum_{\substack{k\geq 1 \\ k|m}} \varphi(k) = \sum_{k\geq 1}\sum_{\substack{m\leq n \\ k|m}} \varphi(k)$$

$$\hookrightarrow \sum_{m=1}^{n}\sum_{\substack{k\geq 1 \\ k|m}} \varphi(k) = \sum_{m=1}^{n} m = 1+2+\cdots+n = \frac{1}{2}n(n+1)$$

$$(1+x)^n = {}^nC_0 1^n + {}^nC_1 1^{n-1}x + {}^nC_2 1^{n-2}x^2 + \cdots + {}^nC_{n-1}1^1 x^{n-1} + {}^nC_n x^n$$

$$= {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_{n-1}x + {}^nC_n x^n$$

$$2^n = (1+1)^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_{n-1} + {}^nC_n$$

$$= \sum_{r=0}^{n} {}^nC_r$$

$$\Rightarrow \sum_{r=0}^{n} {}^nC_r = 2^n$$

Now we wish to find $\sum_{k\geq 0} \binom{1000}{2k} = \sum_{k=0}^{500} \binom{1000}{2k}$

$$(1+1)^{1000} = \sum_{n\geq 0} \binom{1000}{n} \quad -①$$

$$(1-1)^{1000} = \sum_{n\geq 0} \binom{1000}{n}(-1)^n \quad -②$$ $\longrightarrow$ check it once

Adding ① & ② we get,

$$2^{1000} + 0^{1000} = \sum_{n\geq 0}\left(\binom{1000}{n} + \binom{1000}{n}(-1)^n\right)$$

$$\Rightarrow 2^{1000} = \sum_{n\geq 0}\left(\binom{1000}{n}\left(1+(-1)^n\right)\right)$$

$$1+(-1)^n = \begin{cases} 2 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

$$\Rightarrow 2^{1000} = \sum_{k\geq 0}\left(\binom{\cdot}{\cdot} 2\right) + \sum_{k\geq 0}\left(\binom{1000}{2k+1} 0\right)$$

$$\Rightarrow \sum_{k\geq 0}\binom{1000}{2k} = \frac{2^{1000}}{2} = 2^{999}$$

HomeWork

Q> Compute $\sum_{k\geq 0}\binom{1000}{3k}$